

Relaxed Locally Identifying Coloring of graphs

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Abstract

A *locally identifying coloring* (*lid-coloring*) of a graph is a proper coloring such that the sets of colors appearing in the closed neighborhoods of any pair of adjacent vertices having distinct neighborhoods are distinct. Our goal is to study a *relaxed locally identifying coloring* (*rlid-coloring*) of a graph that is similar to locally identifying coloring for which the coloring is not necessary proper. We denote by $\chi_{rlid}(G)$ the minimum number of colors used in a relaxed locally identifying coloring of a graph G .

In this paper, we prove that the problem of deciding that $\chi_{rlid}(G) = 3$ for a 2-degenerate planar graph G is *NP*-complete. We give several bounds of $\chi_{rlid}(G)$ and construct graphs for which some of these bounds are tightened. Studying some families of graphs allows us to compare this parameter with the minimum number of colors used in a locally identifying coloring of a graph G ($\chi_{lid}(G)$), the size of a minimum identifying code of G ($\gamma_{id}(G)$) and the chromatic number of G ($\chi(G)$).

1 Introduction

Let $G = (V, E)$ be a simple undirected finite graph. Let $c : V \rightarrow \mathbb{N}$ be a coloring of the vertices of G . The coloring c is an *identifying coloring* if any pair of vertices u and v satisfies the property (\mathcal{P}): $c(N[u]) \neq c(N[v])$. Observe that if G has two distinct vertices u and v such that $N[u] = N[v]$ then there is no such coloring and we say the vertices u and v are *twins* in G . Note that the coloring is not necessary proper. We define the *identifying chromatic number* of G , and denote by $\chi_{id}(G)$, the minimum number of colors required by an identifying coloring of G . This notion was introduced by Parreau [6]. In order to give a coloring version of the well-known *identifying codes* defined by Karpovsky *et al.* [4]. Parreau [6] gave an upper and a lower bounds of the identifying chromatic number of a free-twin graph. She characterized the free-twin graphs for which the identifying chromatic number is the number of vertices

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of the graph. In [1], the notion of *locally identifying coloring of graphs* (*lid-coloring*) was introduced and defined as follows: for any two adjacent vertices u and v , the coloring c satisfies both the condition (\mathcal{P}) and the condition (\mathcal{Q}) : $c(u) \neq c(v)$. The *locally identifying chromatic number* of G , denoted by $\chi_{lid}(G)$, is the smallest number of colors used in any *lid* coloring of G . Esperet et al. [1] gave several bounds of the locally identifying chromatic number for different families of graphs as planar graphs, some subclasses of perfect graphs and graphs with bounded maximum degree. They also proved that the problem to decide whether a subcubic bipartite graph with large girth is 3 – *lid*-colorable is an *NP*-Complete problem. An upper bound for any graph was given in term of the maximum degree [2] and the same authors also gave a bound of χ_{lid} for chordal graphs in term of the maximum degree and the chromatic number. Gonçalves et al. [3] proved that for any graph class of bounded expansion, the *lid*-chromatic number is bounded. They also showed that χ_{lid} is bounded for any class of minor closed classes of graphs and they gave an explicit upper bound for χ_{lid} of planar graphs.

Consider now a coloring of a graph G satisfying only the condition (\mathcal{P}) for any pair of adjacent vertices. We obtain the notion of *relaxed locally identifying coloring* (*rlid-coloring*) of a graph, on which we focus in this paper. Define the relaxed locally identifying chromatic number of a graph G , $\chi_{rlid}(G)$, as the smallest number of colors used in a relaxed locally identifying coloring.

Note that if G contains twins u and v we have $c(N[u]) = c(N[v])$. One may ask which influence have twins for *rlid*-coloring?

To answer this question, let \mathcal{R} be an equivalence relation defined as follows: for all vertices $u, v \in V(G)$, we have $u\mathcal{R}v$ if and only if $N[u] = N[v]$. Denote by G/\mathcal{R} the maximal twin-free subgraph of G and let $t(G)$ represent the number of equivalence-classes having at least two vertices in G .

Theorem 1. *Let G/\mathcal{R} be a maximal twin-free subgraph of a connected graph G . Then, we have $\chi_{rlid}(G/\mathcal{R}) - t(G) \leq \chi_{rlid}(G) \leq \chi_{rlid}(G/\mathcal{R})$.*

Proof. Consider a *rlid*-coloring c of G/\mathcal{R} , and prove that c also is a *rlid*-coloring of G . For each vertex x and its twin y in G , put $c(x) = c(y)$. Since in G , we do not interest to distinguish the twins then c defines a *rlid*-coloring of G .

Now, prove the second inequality. Let c be a *rlid*-coloring of G . Consider the coloring c' defined as follows: $c'(u) = c(u)$ if the vertex u has no twin in G and color $t(G)$ other vertices of G/\mathcal{R} with different colors $\chi_{rlid}(G) + 1$ until $\chi_{rlid}(G) + t(G)$. This coloring gives a *rlid*-coloring of G/\mathcal{R} . \square

Note that if G is twin-free then $\chi_{rlid}(G) = \chi_{rlid}(G/\mathcal{R})$. In Section 4, we exhibit an example for which the upper bound is tighten. In this paper, we are interested in a studying χ_{rlid} of a graph G . If G contains twins, we are interested in separating all pairs of adjacent vertices except twins in terms of G/\mathcal{R} . We give several bounds of the relaxed locally identifying chromatic number for some subclasses of graphs and compare χ_{rlid} with both χ_{lid} and χ (the chromatic number).

Our starting result is given as follows:

Theorem 2. *Let G be a graph of order n and G/\mathcal{R} be a maximal twin-free subgraph of G .*

Then we have $\log \omega(G/\mathcal{R}) + 1 \leq \chi_{rlid}(G) \leq n$.

Where $\omega(G)$ represents the maximal size of a clique of G .

We will prove this theorem in Section 4. In Section 3, we prove that the problem of deciding that $\chi_{rlid}(G) = 3$ is *NP*-complete for a connected 2-degenerate planar graph G without twins and it is polynomial for a bipartite graph. In Section 4, we start by proving that the lower bound of Theorem 1 is tight. We characterize graphs G satisfying $\chi_{rlid}(G) = n$. We show the lower bound of Theorem 2 and exhibit a family of graphs for which this bound is attained. We also study the split graphs for which we give an upper and a lower bound of χ_{rlid} , and we construct two graphs which tighten these bounds.

So, this paper is structured as follows: the next section presents basic definitions used in this paper. Then in Section 3, we start by studying the complexity of this problem and we show that non trivial bipartite graphs are *3-rlid*. Further Section 4 is spent to establish relationship between χ_{id} , γ_{id} and χ_{rlid} . Section 5 is spent to study the split graphs. We give an upper and a lower bound for these graphs and we exhibit a graph for which the lower bound is tight. Finally, we conclude by some remarks and some open questions.

2 Useful definitions

Let $G = (V, E)$ be a finite connected graph, where V (we also write $V(G)$) is the vertex set and E (we also write $E(G)$) is the edge set. We denote by $N(x)$ (resp. $N[x]$) the open (resp. closed) neighborhood of x , the set of all adjacent vertices of x and we have $N[x] = N(x) \cup \{x\}$.

A vertex x is a twin of another vertex y if we have $N[x] = N[y]$. A graph G is called *twin-free* if G contain no twin. The symmetric difference of two vertices x and y , denoted by $N[x] \triangle N[y]$, is the set of vertices $(N[x] \setminus N[y]) \cup (N[y] \setminus N[x])$. The maximal size of a clique in a graph G is denoted by $\omega(G)$. A *universal vertex* of a graph is a vertex adjacent to all the others. A planar graph is a graph which can be drawn in the plane without any edges crossing. Let k be an integer, a graph G is *k-degenerate* if every subgraph of G has a vertex of degree at most k .

A subset C of vertices of G is an *identifying code* of G if C is a dominating set of G (i.e. for each vertex $v \in V(G)$, we have $N[v] \cap C \neq \emptyset$) and C is a separating set of G (i.e. for each pair of distinct vertices $u, v \in V(G)$, $N[u] \cap C \neq N[v] \cap C$). We denote by $\gamma_{id}(G)$ the minimum cardinality of an identifying code of G .

Since an *id*-coloring and a *lid*-coloring are *rlid*-coloring, we have trivially

$$\chi_{rlid}(G) \leq \chi_{id}(G) \text{ and } \chi_{rlid}(G) \leq \chi_{lid}(G).$$

Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, $G_1 \bowtie G_2$ is the *join graph* of G_1 and G_2 , in which the vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2 \cup \{v_1 v_2 \mid v_1 \in V_1, v_2 \in V_2\}$.

Give $p \geq 2$ an integer, we define the graph $H_p = (V, E)$ from a clique K of size 2^p , and three stable sets S_1, S_2, S_3 of size p as follows (See Figure. 1).

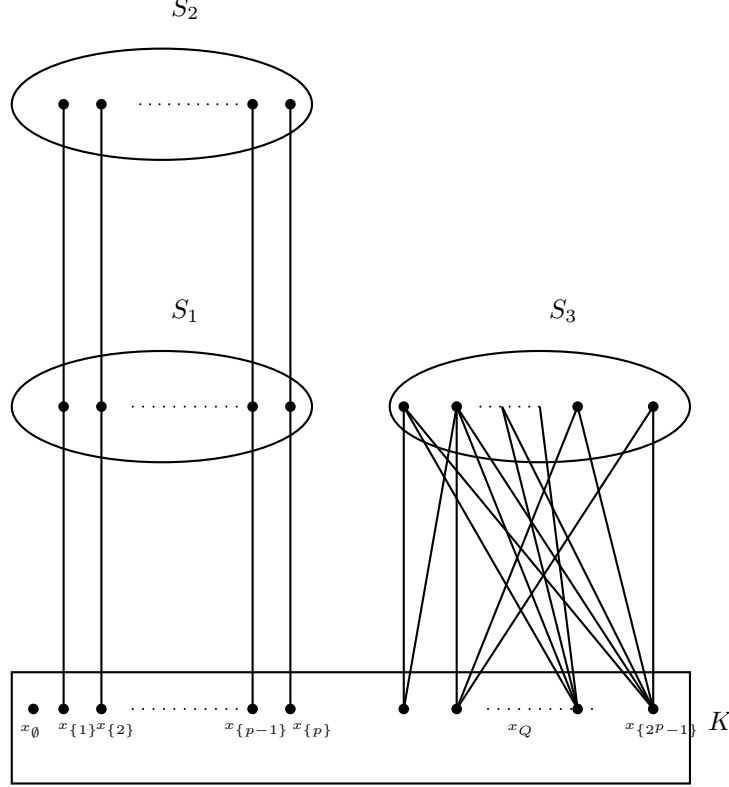


Figure 1: The graph H_p and its components K, S_1, S_2 and S_3

$K = \{x_Q, Q \in \mathcal{P}(p)\}$: it means that Q is a subset of $\{1, 2, \dots, p\}$.
 $S_1 = \{y_1, y_2, \dots, y_p\}$, $S_2 = \{y'_1, y'_2, \dots, y'_p\}$ and $S_3 = \{z_1, z_2, \dots, z_p\}$.

The edges are defined by:

- for all $i \in [p]$, $x_{\{i\}}y_i \in E$ and $y_iy'_i \in E$.
- and for all $Q \subseteq \mathcal{P}[p]$ with $|Q| \geq 2$, $x_Qz_i \in E$ iff $i \in Q$.

For an integer $k \geq 1$, we denote by P_{2k}^{k-1} the $k - 1$ power of the path on $2k$ vertices.

Given a graph G , we construct a graph G^* by replacing each edge of G by a path of length 3, and adding to each vertex of G a pendant vertex (See Figure. 2).

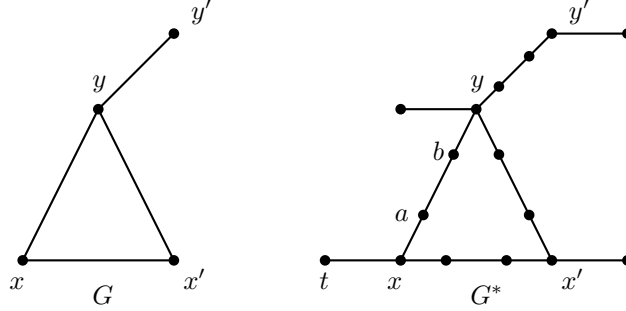


Figure 2: The graph G^* associated to the graph G where each edge in G is replaced by a path of length 3 in G^* and a pendant vertex is attached to each vertex of G .

3 Complexity results

This section is devoted to study the problem of complexity. Before we need to prove the following result which gives a necessary and sufficient condition for the graph G^* (see Figure. 2) to have a k -rlid-coloring:

Theorem 3. *Let $k \geq 3$ be an integer and $G = (V, E)$ be a connected twin-free graph. Then G is k -colorable if and only if G^* is k -rlid-colorable.*

Proof. Let $k \geq 3$ be an integer and G be a connected twin-free graph of order $n \geq 3$. First, prove that if G admits a k -coloring then G^* is k -rlid-colorable.

Let c be a k -coloring of G and G^* be the graph associated to G .

Define a coloring c' of G^* from c as follows:

- $c'(x) = c(x)$ for any vertex $x \in V(G)$;
- for a pendant vertex t adjacent to x , $c'(t) = c(y)$ where y is any vertex of G adjacent to x .
- for $xy \in E(G)$, let $xaby$ be the corresponding path in G^* . Since $k \geq 3$, choose a color $q \neq c(x)$ and $q \neq c(y)$. Fix $c'(a) = c'(b) = q$.

We claim that c' is a $rlid$ -coloring of G^* .

According to the previous notation, we only have to check edges xt or xa or ab . Remark that for $x \in G$, $y \in G$ with $xy \in E(G)$, $xaby$ is the corresponding path in G^* and $c'(t) = c(y)$. We have $c'(N[x])$ contains $\{c(y), c(x), c'(a)\}$, therefore $|c'(N[x])| \geq 3$. Since t is a leaf, we have $|c'(N[t])| \leq 2$. Thereover by definition of c' , for any subdivision $x'a'b'y'$ we have $c'(a') = c'(b')$ then $|c'(N[a'])| \leq 2$. Thus extremities of xt or xa edges are identified. In any path $xaby$, $c'(N[a]) = \{c'(a), c(x)\}$, $c'(N[b]) = \{c'(a), c(y)\}$. Since $c(x) \neq c(y)$ by definition of a coloring c in G and since $xy \in E(G)$, we obtain that $c'(N[a]) \neq c'(N[b])$.

Let c be a k -rlid-coloring of G^* . To achieve the proof it is enough to show that $c(x) \neq c(y)$ for any edge $xy \in E(G)$. If $c(x) \neq c(y)$ then let $xaby$ be

the path of G^* corresponding to edge xy in G . If $c(x) = c(y)$ then $c(N[a]) = \{c(a), c(x), c(b)\} = c(N[b])$, a contradiction. \square

For planar graphs, we have the following result

Theorem 4. [5] *The problem of 3-colorability of a planar graph is NP-complete.*

Let G be a planar graph, and G^* be the graph associated to G . Then G^* is also planar. On the other hand, G^* is 2-degenerate (by construction). By Theorem 3, G is 3-colorable iff G^* is 3-*rlid*-coloring. By Theorem 4, the problem of deciding if a planar graph G is 3-colorable is NP-complete, then we deduce that the problem of deciding that G^* is 3-*rlid*-coloring is also NP-complete. Then, we obtain the following result:

Corollary 5. *The problem of deciding that a 2-degenerate planar graph is 3-*rlid*-coloring is NP-Complete.* \square

In [1], it was shown that:

Theorem 6. [1] *For any fixed integer g , deciding whether a bipartite graph with girth at least g and maximum degree 3 is 3-*lid*-colorable is an NP-complete problem.*

Thanks to Corollary 5, *rlid*-coloring is still NP-complete for 2-degenerate planar graphs. However, we will show that it is polynomially solvable for bipartite graphs.

Theorem 7. *Let G be a bipartite graph of order at least 3, then $\chi_{rlid}(G) \leq 3$.*

Proof. Let G be a bipartite graph. We have two cases: G contains a universal vertex or no.

- **G contains a universal vertex.**

Then $G \simeq K_{1,p}$ with $p > 1$ then:

- If $p = 2$. Since $K_{1,2} \simeq P_3$, one may assume that $\chi_{rlid}(K_{1,2}) \leq 3$.
- If $p > 2$: Let us prove that $\chi_{rlid}(K_{1,p}) \leq 3$. Assume that u is the universal vertex and the other vertices are denoted by $\{v_1, v_2, \dots, v_p\}$. The coloring c defined by $c(u) = 1$, $c(v_1) = 2$ and $c(v_i) = 3$ for all $i = 2, \dots, p$ is a *rlid*-coloring.

- **G does not contain an universal vertex.**

Let x be a vertex of G . Considering the partition of vertices of G into levels L_0, L_1, \dots, L_p according to the vertex x , we have $L_0 = \{x\}$ and $L_j = \{v/d(x, v) = j\}$ for $j \geq 1$. Since G is a finite graph, we have a finite number of levels. Let A_i and B_i be a subdivision of L_i in two disjoint subsets such that $v \in A_i$ iff $N(v) \cap L_{i+1} = \emptyset$ and $N(v) \cap L_{i-1} = B_{i-1}$, and $v \in B_i$ iff $N(v) \cap L_{i+1} = L_{i+1}$ and $N(v) \cap L_{i-1} = B_{i-1}$.

Let v be a vertex of G ($v \neq x$). Consider the coloring c of G with three colors $\{1, 2, 3\}$ as follows, for all $i \geq 0$:

$$c(v) = \begin{cases} 1 & \text{if } v \in L_{4i} \\ 1 & \text{if } v \in A_{4i+1} \\ 2 & \text{if } v \in B_{4i+1} \\ 3 & \text{if } v \in L_{4i+2} \\ 3 & \text{if } v \in A_{4i+3} \\ 2 & \text{if } v \in B_{4i+3} \end{cases}$$

Since G is a bipartite graph, then we can not have edges between vertices in a same level. Observe that L_2 is not empty.

For all $i \geq 0$, remark that for $v \in L_{4i}$, $c(N[v]) = \{1, 2\}$.

When $v \in A_{4i+1}$, we get $c(N[v]) = \{1\}$, if $v \in B_{4i+1}$ we have $c(N[v]) = \{1, 2, 3\}$.

If $v \in L_{4i+2}$, we obtain $c(N[v]) = \{2, 3\}$. Finally in L_{4i+3} , if $v \in A_{4i+3}$ we get $c(N[v]) = \{3\}$, if $v \in B_{4i+3}$ we obtain $c(N[v]) = \{1, 2, 3\}$.

Let u and v be two adjacent vertices. Since G is a bipartite graph, then u and v do not belong to a same level. Suppose that $u \in L_i$, then we have either $v \in L_{i-1}$ or $v \in L_{i+1}$ and suppose that $N[u] \neq N[v]$. We have to consider four cases:

• **Case 1.** $u \in L_{4i}$

If $u \in L_{4i}$ and $v \in B_{4i-1}$, we have $c(N[u]) \triangle c(N[v]) = \{3\}$.

If $u \in B_{4i}$, then $v \in L_{4i+1}$ and $c(N[u]) \triangle c(N[v]) = \{2 \text{ or } 3\}$.

• **Case 2.** $u \in L_{4i+1}$

If $u \in A_{4i+1}$, if $v \in B_{4i}$ we have $c(N[u]) \triangle c(N[v]) = \{2\}$.

If $u \in B_{4i+1}$, then either $v \in B_{4i}$ and $c(N[u]) \triangle c(N[v]) = \{3\}$ or $v \in L_{4i+2}$ and we get then $c(N[u]) \triangle c(N[v]) = \{1\}$

• **Case 3.** $u \in L_{4i+2}$

If $u \in L_{4i+2}$ and $v \in B_{4i+1}$, we obtain $c(N[u]) \triangle c(N[v]) = \{1\}$.

If $u \in B_{4i+2}$, then either $v \in A_{4i+3}$ and $c(N[u]) \triangle c(N[v]) = \{2\}$ or $v \in B_{4i+3}$ and $c(N[u]) \triangle c(N[v]) = \{1\}$.

• **Case 4.** $u \in L_{4i+3}$

If $u \in A_{4i+3}$, if we have $v \in B_{4i+2}$ then $c(N[u]) \triangle c(N[v]) = \{2\}$.

If $u \in B_{4i+3}$, then either $v \in B_{4i+2}$ and $c(N[u]) \triangle c(N[v]) = \{1\}$ or $v \in L_{4(i+1)}$ and $c(N[u]) \triangle c(N[v]) = \{3\}$.

In all case, observe that u and v are distinguished. Then we have $\chi_{rlid}(G) \leq 3$.

□

Remark that there is no graph G with $\chi_{rlid}(G) = 2$. Moreover, a graph is $1 - rlid$ -coloring iff it is the disjoint union of cliques. Therefore, our proof of Theorem 7 provides a polynomial time algorithm for $3 - rlid$ -coloring graph if it is bipartite.

4 Relationship between γ_{id} , χ_{id} and χ_{rlid}

The notion of *identifying chromatic number* χ_{id} and *locally identifying chromatic number* χ_{lid} were given for twin-free graphs, by as against for the *relaxed locally identifying chromatic number*, we even study the graphs that contain twins. In the following, we show that the lower bound of inequality of Theorem 1 is tight.

Property 8. *Let $p \geq 4$ be an integer and $t = \binom{p-1}{2}$. There is a graph G such that $\chi_{rlid}(G) = p$ and $\chi_{rlid}(G) = \chi_{rlid}(G/\mathcal{R}) - t$.*

Proof.

Construct the graph G such that $\chi_{rlid}(G) = p$ and $t = \binom{p-1}{2}$. Consider K_{p+t} with $\{x_1, x_2, \dots, x_t, \dots, x_{p+t}\}$ the set of its vertices, K_{p+t}^* is the graph associated to K_{p+t} as defined in Section 2. Denote by z_i the pendant vertex of x_i for $i = 1, \dots, p+t$ and x_i^j and x_j^i are the vertices subdivide the edge $x_i x_j$ with x_i^j (resp. x_j^i) is adjacent to x_i (resp. to x_j) for $i, j = 1, \dots, p+t$. The graph G is obtained from K_{p+t}^* by adding t twins y_1, y_2, \dots, y_t respectively to x_1, x_2, \dots, x_t . First, observe that $G/\mathcal{R} \simeq K_{p+t}^*$. Then $\chi_{rlid}(G/\mathcal{R}) = p+t$.

Now, we will prove that G admits a p -*rlid*-coloring.

Let c be a coloring of G defined as follows: put $c(x_{t+i}) = i$ for $i \geq 1$. For the vertices belonging to the same equivalence-class, c is defined by one to one mapping $\{(c(x_i), c(y_i)) \mid 1 \leq i \leq t\} \mapsto \mathcal{P}_2(p-1)$. The vertices x_i^j and x_j^i receive the same color p for all (i, j) except $(p, t+1)$ and we put $c(x_p^{t+1}) = c(x_{t+1}^p) = p-1$. For the leaf z_i adjacent to x_i , put $c(z_i) = 1$ if $i > t$ and $c(x_i) \neq 1$ and if $c(x_i) = 1$, z_i receives the color 2. Put $c(z_i) \in \{1, \dots, p-1\} \setminus \{c(x_i), c(y_i)\}$ if $i \leq t$.

Let u and v be two adjacent vertices of G (we are not interested to distinguish two vertices belonging to a same equivalence-class).

Observe that if $u = x_i^j$ and $v = x_j^i$ with $1 \leq i, j \leq t$, we have $c(z_i) \in c(N[x_i^j])$ and $c(z_i) \notin c(N[x_j^i])$. If $t+1 \leq i, j \leq p+t$, we have $c(x_i) = i \in c(N[x_i^j])$, $c(x_j) = j \notin c(N[x_i^j])$ and $c(x_i) \neq c(x_j)$. If $1 \leq i \leq t$ and $t+1 \leq j \leq p$, remark that $|c(N[x_i^j])| = 3$ and $|c(N[x_j^i])| = 2$. If $u = x_i$ and $v = x_j^i$, then $c(z_i) \in c(N[x_i])$ with $c(z_i) \notin c(N[x_j^i])$ if $1 \leq i \leq t$ and $|c(N[x_i^j])| = 2$ and $|c(N[x_i])| = 4$ if $t+1 \leq i \leq p+t$.

If $u = x_i$ and $v = z_i$, then if $1 \leq i \leq t$, we get $|c(N[x_i])| = 4$ and $|c(N[z_i])| = 3$.

If $t+1 \leq i \leq t+p$, we have $|c(N[x_i])| = 3$ and $|c(N[z_i])| = 2$.

Then, each two adjacent vertices of G are distinguished by c . \square

Theorem 7 claims that bipartite graphs admit 3-*rlid*-coloring despite χ_{id} of bipartite graph is not bounded. However in [1], it is shown the following result:

Theorem 9. [1] *Let G be a free-twin graph. Then $\chi_{id}(G) \leq \gamma_{id}(G) + 1$.*

For *rlid*-coloring, this gives:

Property 10. *For any twin-free graph G , we have $\chi_{rlid}(G) \leq \gamma_{id}(G) + 1$. \square*

Our goal is to use this inequality to characterize graphs G satisfying $\chi_{rlid}(G) = n$. A characterization of graphs for which the id -chromatic number equals the order of G is given in [6].

Theorem 11. [6] *Given a connected twin-free graph G , we have $\chi_{id}(G) = n$ if and only if G is a complete graph minus maximal matching or $G = K_1 \bowtie \mathcal{H}$ where $\mathcal{H} = \mathcal{H}_1 \bowtie \dots \bowtie \mathcal{H}_l$ with $\mathcal{H}_i \simeq P_{2k}^{k-1}$ or $\mathcal{H}_i = \overline{K_2}$ for $i = 1, \dots, l$, $k \geq 2$.*

For a relaxed locally identifying coloring we obtain:

Corollary 12. *Let G be a connected twin-free graph of order n . Then, $\chi_{rlid}(G) = n$ if and only if $G \simeq K_1 \bowtie \mathcal{H}$ with $\mathcal{H} = \mathcal{H}_1 \bowtie \mathcal{H}_2 \bowtie \dots \bowtie \mathcal{H}_l$ and $\mathcal{H}_i \simeq P_{2k}^{k-1}$ or $\mathcal{H}_i = \overline{K_2}$ for $i = 1, \dots, l$.*

Proof. Let G be a connected twin-free graph of order $n \geq 3$.

The proof of **if part**.

If $G \simeq K_1 \bowtie \mathcal{H}$ where $\mathcal{H} = \mathcal{H}_1 \bowtie \mathcal{H}_2 \bowtie \dots \bowtie \mathcal{H}_l$ with $\mathcal{H}_i \simeq P_{2k}^{k-1}$ or $\mathcal{H}_i = \overline{K_2}$ and $i = 1, \dots, l$, let u be the universal vertex corresponding to K_1 and $\{v_1, v_2, \dots, v_{2k}\}$ the set of vertices of \mathcal{H}_1 for which $\mathcal{H}_i \simeq P_{2p}^{k-1}$. Let c be an $rlid$ -coloring of G .

For all s such that $2k > s > k$, we have $N[v_s] \triangle N[v_{s+1}] = \{v_{s-k}\}$. Hence for all $j, t \leq k$, $c(v_j) \neq c(v_t)$. Moreover, for all $s > k$, we have $N[v_s] \supseteq \{v_{k+1}, \dots, v_{2k}, u\} \cup \bigcup_{t \geq 2} V(\mathcal{H}_t)$.

Hence $\forall j \leq k$, we have $c(v_j) \notin c(\{v_{k+1}, \dots, v_{2k}, u\} \cup \bigcup_{t \geq 2} V(\mathcal{H}_t))$. Otherwise,

$$c(N[v_{j+k-1}]) = c(N[v_{j+k}]).$$

By symmetry, for any $j > k$, we have $c(v_j) \notin c(V \setminus v_j)$.

Similarly, we obtain the same results for each \mathcal{H}_j with $j \geq 2$.

If $\mathcal{H}_1 \simeq \overline{K_2}$, let a and b be the vertices corresponding to $\overline{K_2}$. We get $\{b\} = N[a] \triangle N[u]$. Moreover $N[a] = V \setminus \{b\}$, then b has a color different to all colors of other vertices of graph. We obtain a similar result for the vertex a .

Finally all vertices of G have different colors.

The proof of **only if part**.

If $\chi_{rlid}(G) = n$ then χ_{rlid} -coloring is χ_{id} -coloring and by Theorem 11 [6] we have the result. \square

Corollary 12 shows that $\chi_{rlid}(G) = \chi_{id}(G)$ when $\chi_{rlid}(G) = n$ then we also have $\chi_{rlid}(G) = \chi_{lid}(G)$.

Now, we can show that χ_{rlid} may be very small compared with χ_{lid} , for this we will show the lower bound of Theorem 2. We will exhibit a family of graph for which this bound is tight.

Lemma 13. *Let G be a graph and G/\mathcal{R} be the maximal free-twin subgraph of G . Then $\chi_{rlid}(G) \geq \log(\omega(G/\mathcal{R})) + 1$.*

Proof. Let K an ω -clique of G/\mathcal{R} . Let $V' = V(G) \setminus K$ and c be a *rlid*-coloring of G .

Since for each pair of distinct adjacent vertices x, y in K , we must have $c(N[x]) \neq c(N[y])$ (because c is an *rlid*-coloring), we also have for each vertex $x \in K$, $c(K)$ belongs to $c(N[x])$, since x has in its neighborhoods all vertices of K and some vertices of V' .

Thus, for each two distinct adjacent vertices $x, y \in K$, we have $c(N[x]) \neq c(N[y])$ iff $c(N[x]) - c(x) \neq c(N[y]) - c(y)$.

Let $p = |c(V) \setminus (c(K))|$. With p distinct colors, we can distinguish at most 2^p vertices. Thus $2^p \geq |K|$. Hence $p \geq \log(|K|)$. Then, the total number of different colors used by c is at least $|c(K)| + p \geq p + 1$. \square

Now we exhibit a family of graphs satisfying the equality:

Corollary 14. *If $p \geq 2$, then $\chi_{rlid}(H_p) = \log \omega(H_p) + 1$.*

Proof. We define a *rlid*-coloring c as follows:

- $c(x_Q) = p + 1$ for all $Q \subseteq \mathcal{P}([p])$
- $c(y_i) = c(z_i) = i$ for all $i \in [p]$.
- $c(y'_i) = i + 1p \text{ mod } p$ for all $i \in [p]$.

Now, we have $c(N[x_Q]) = Q \cup \{p + 1\}$ for all $Q \subseteq \mathcal{P}([p])$, $c(N[z_i]) = \{i, p + 1\}$ for each $i \in [p]$, $c(N[y'_i]) = \{i, i + 1 \text{ mod } p\}$ for each i and $c(N[y_i]) = \{i, i + 1 \text{ mod } p, p + 1\}$ for each i . Since there is no edge between any $y \in S_1$ and any $x_Q \in K$ with $|Q| \geq 2$ and no edge between any $z \in S_3$ and any $i_{\{i\}}$ for any $i \in [p]$, then c defines a *rlid*-coloring. \square

From Corollary 14, we have $\chi_{rlid}(H_p) = p + 1$ despite $\chi_{lid}(H_p) \geq \chi(H_p) \geq \omega(H_p) = 2^p$.

5 Split graphs

Let $G = (K \cup S, E)$ be a split graph with $K = \{x_1, x_2, \dots, x_k\}$ a clique of maximal size $\omega(G) = k$ and $S = \{s_1, s_2, \dots, s_p\}$ a stable of G . For this class of graphs, the χ_{rlid} is given by the following theorem :

Theorem 15. *Let $G = (K \cup S, E)$ be a connected split graph without twins. Then, $\log(\omega(G)) + 2 \leq \chi_{rlid}(G) \leq \omega(G) + 2$.*

Proof. Let us prove that $\chi_{rlid}(G) \geq \log(\omega(G)) + 2$.

In order to identify all vertices of K , we have $\omega(G) \leq 2^p$. Let c be a *rlid*-coloring of G .

According to the proof of Theorem 13, we have $|c(V) \setminus c(K)| \geq p$. Let us prove that we need to add at least two colors to color K .

If $|c(K)| \geq 2$, then we are done.

If $\omega < 2^p$, then we are done.

Therefore, we can assume that $\omega(G) = 2^p$ and $|c(K)| = 1$. For any vertex $x \in K$ put $c(x) = p + 1$. Let $u \in S$ and denote by $I(u)$ the set of colors appearing

in $N[u]$. Observe that $I(u)$ contains the color $p + 1$. If u is adjacent to some vertices in K . Since $\omega(G) = 2^p$ then there exists a vertex $x \in K$ such that $c(N[x]) = \{i, p + 1\}$ with $i = c(v)$ and $v \in S$. This shows that v is the vertex belonging to S , adjacent to x which gives $c(N[x]) = c(N[v])$, a contradiction. Then in this case, we need to add at least two colors in K and we are done.

Now, let us prove $\chi_{rlid}(G) \leq \omega(G) + 2$. One may assume that G is a connected graph and K is a maximal clique of G of size ω . Start by proving that $k - 1$ colors in S suffice to distinguish all vertices in K . Inductively, we construct a subset S' of S with $|S'| \leq k - 1$ (where $k = |K|$) such that for any two distinct vertices x and y in K , we have $N[x] \cap S' \neq N[y] \cap S'$.

Choose any vertex $u \in S$. Since K is maximal then $K \cap N[u] \neq \emptyset$. Denote $K_1 = N[u] \cap K$ and $K_2 = K \setminus K_1$. Let G_i ($i = 1, 2$) be the split graph induced by K_i and S_i defined by $v \in S_i$ iff $v \in S$ and $N[v] \subseteq K_i$. Let us prove that G_i , $i = 1, 2$, is twin-free and that K_i ($i = 1, 2$) is maximal.

By construction, K_i ($i = 1, 2$) is maximal.

Now suppose that $x, y \in G_i$ are twins. Since K_i is maximal then $x, y \in K_i$. By definition of G_i , any vertex $x_i \in V(G) \setminus V(G_i)$ is either adjacent or not to x and y . Therefore, x and y will be twins in G , which yields a contradiction. Now, apply the induced hypothesis in G_1 and G_2 . We get $|S'| \leq k - 1$.

So set $S' = S_1 \cup S_2$ and let $c(s_i) = i$ for any vertex $s_i \in S'$. Since $N[x] \cap S' \neq N[y] \cap S'$ for all two distinct vertices $x, y \in K$ then there is at most one vertex namely $u \in K$ such that $N[u] \cap S' = \emptyset$. Now, partition K in K_1 and K_2 such that any vertex $x \in K_1$ satisfies $|N[x] \cap S'| = 1$ and any vertex $y \in K_2$ satisfies $|N[y] \cap S'| \geq 2$. Let c be a *rlid*-coloring of G .

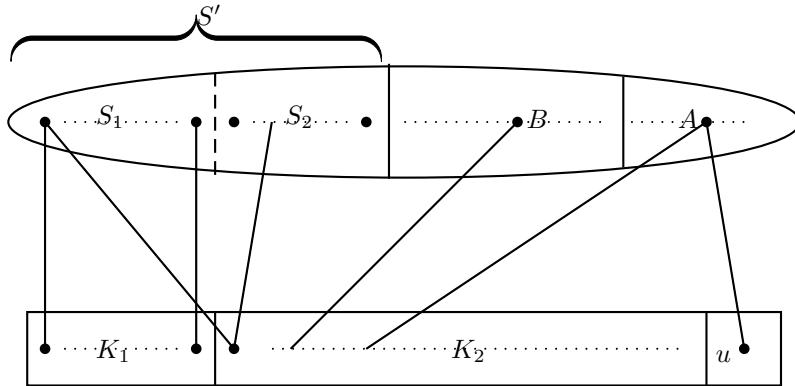


Figure 3: The split graph with its components K and S .

Case 1 : there exists $u \in K$ such that $N[u] \cap S \neq \emptyset$.

Let $A = N[u] \cap S$ and $B = S \setminus (S' \cup A)$ (see Figure.3).

Subcase 1.1. $K_1 \neq \emptyset$ and $A = \emptyset$. Consider the following coloring: assign to all vertices in $B \cup K_1 \cup K_2$ the color k and put $c(u) = k + 1$. Let $x \in K$ and $s \in S$ be two vertices in G . We know that all vertices of K are distinguished by S' and since $k + 1 \in c(N[x]) \setminus c(N[s])$ then x and s are distinguished.

Subcase 1.2. $K_1 \neq \emptyset$ and there exists $y \in K_1$ such that $N[y] \cap A \neq \emptyset$. We define a coloring c as follows: all vertices of $K_1 \setminus \{y\} \cup K_2 \cup (S \setminus S')$ receive the color k , put $c(y) = k + 1$ and $c(u) = k + 2$.

We claim that any two adjacent vertices of G are discriminated. Indeed, we know that all vertices of K are distinguished by S' . Now let us check the edges between K and S . Let $x \in K$ and $s \in S$ be two adjacent vertices. Suppose that $N[x] = N[s]$.

- If $x \in K_1$, since $k + 2 \in c(N[x])$ then $s \in A$. Thus by assumption, s and x are not adjacent.
- If $x \in K_2$, since $k + 1$ belongs to $c(N[x])$ then $s \in B \cup S'$. Thus $k + 2$ belongs to $c(N[x]) \cap c(N[s])$.
- If $x = u$, then $s \in A$. Thus $k + 1 \in c(N[x]) \cap c(N[s])$, which yields a contradiction.

Subcase 1.3. $K_1 \neq \emptyset$ and for all $x \in K_1$ we have $N[x] \cap A \neq \emptyset$.

- If $A = \{v\}$, then there exists $y \in K$ not adjacent to v . Put $c(v) = k$, $c(u) = k + 2$, $c(y) = k + 1$ and for all $z \in (K \setminus \{y\}) \cup B$, put $c(z) = k$.

If $x \in K$ and $s \in S \setminus A$, since $k + 2 \in c(N[x]) \setminus c(N[s])$ then x and s are distinguished.

Additionally, the pair $x \in K$ and $s = v$ are also distinguished since $k + 1 \in c(N[x]) \setminus c(N[v])$.

- If $|A| \geq 2$, consider the following coloring: the vertices of $B \cap K$ receive the color k . There exists a vertex $w \in A$ such that $c(w) = k + 1$ and $\forall v \in A \setminus \{w\}$, $c(v) = k + 2$.

Let $x \in K$ and $s \in S$ be two vertices.

- If $s \notin S'$ and $x \neq u$, then the color $k + 1 \in c(N[x]) \setminus c(N[s])$.
- If $s \in S' \cup B$ and $x = u$, the vertices x and s are not adjacent.
- If $s \in A$ and $x = u$, one of the colors $k + 1$ or $k + 2$ belongs to $c(N[s])$ but both are in $c(N[u])$.
- If $s \in S'$ and $x \in K_2$, there exists a vertex $s' \in S' \setminus \{s\}$ such that $c(s') \neq c(s)$. We have $c(s_j) \in c(N[x]) \setminus c(N[s])$.
- If $x \in K_1$ and $s \in S'$, either the color $k + 1$ or $k + 2$ belongs to $(N[x]) \setminus c(N[s])$.

Case 2 : The vertex u does not exist in K . In this case $A = \emptyset$ and :

- If $K_1 = \emptyset$, consider the following coloring: all vertices in $K \cup B$ receive the color k .

Let $x \in K$ and $s \in S$ be two vertices in G . Then:

- If $s \in S'$, since $|N[x] \cap S'| \geq 2$ then there exists a vertex $s' \in S'$ such that $s \neq s'$ and $c(s') \in c(N[x]) \setminus c(N[s])$.
- If $s \in S \setminus S'$, the vertex x is adjacent to at least two vertices in S' .
- $K_1 \neq \emptyset$. If there exists $x \in K$ such that $|N[x] \cap S_1| = 1$.

There exists a vertex $y \in K_2$ which is not adjacent to $s_1 \in S_1$. Consider the

following coloring: put $c(x_1) = k + 1$ (x_1 is the vertex belonging to K_1 and has only s_1 as neighbor in S) and $c(y) = k + 2$. All vertices in $B \cup K \setminus \{x_1, y\}$ receive the color k .

Let $x \in K$ and $s \in S$ be two vertices in G :

- **If** $x \in K_1 \setminus \{x_1\}$ **and** $s \in S'$, then $k + 1 \in c(N[x]) \setminus c(N[s])$.
- **If** $x = x_1$ **and** s_1 , we have $k + 2 \in c(N[x]) \setminus c(N[s])$.
- **If** $x \in K_1$ **and** $s \in S \setminus S'$, there exists a color $i = c(s_i)$ with $c(s_i) \in S'$ such that $i \in c(N[x]) \setminus c(N[s])$.
- **If** $x \in K_2$ **and** $s \in S'$, the color $j \in c(N[x]) \setminus c(N[s])$ with $j = c(s_j)$, $s_j \in S'$ and $s_j \neq s$.
- **If** $x \in K_2$ **and** $s \in S \setminus S'$, the vertex x has at least two neighbors in S' . \square

In [1], the following result is given:

Theorem 16. [1] Let $G = (K \cup S, E)$ be a split graph.
If $\omega(G) \geq 3$ or G is a star, then $\chi_{lid}(G) \leq 2\omega(G) - 1$.

In the following, we give two split graphs $Q_1(p)$ and $Q_2(p)$ (with $p \geq 2$) such that the lower bound of Theorem 15 is tight for the first, and the upper bound of Theorem 3 is attained for the second graph.

The first graph $Q_1(p)$ is a split graph where $K = \{x_Q, Q \in \mathcal{P}(p)\}$ with Q is a subset of $\{1, \dots, p\}$ and the stable $S = \{s_1, s_2, \dots, s_{p-1}\}$. The size of K is 2^{p-1} . The second graph $Q_2(p)$ is a split graph where the set-vertex of K is $\{v_1, v_2, \dots, v_{p-1}, v_p\}$ and the set-vertex of S is $\{s_1, s_2, \dots, s_{p-1}\}$. The edges are defined by:

- $v_i s_i$ is an edge for $i = 1, \dots, p - 1$;
- $v_i v_j$ is an edge for $1 \leq j \leq p$.

Proposition 17. For $p \geq 2$, we have $\chi_{rlid}(Q_1(p)) = \log(\omega(Q_1(p))) + 2$.

Proof. Let c be a $rlid$ -coloring of $Q_1(p)$. Suppose that there exist two vertices s_i and s_j such that $c(s_i) = c(s_j)$ for $1 \leq i, j \leq p - 1$, $i \neq j$ and $N[x_Q] \triangle N[x_{Q'}] = \{s_i, s_j\}$, then we obtain $c(N[x_Q]) = c(N[x_{Q'}])$, a contradiction. Then, all s_j ($1 \leq i \leq p - 1$) have different colors.

Let x_Q and $x_{Q'}$ be two vertices of K such that $|Q| > |Q'|$. We have $|N[x_Q]| > |N[x_{Q'}]|$. In K , if there exists a vertex x_Q having a unique neighbor $s_j \in S$ ($j = 1, \dots, p - 1$), then $N[x_Q] \triangle N[s_j] = \{x_\emptyset\}$, which implies that $c(x_\emptyset) \notin c(K \setminus \{x_\emptyset\} \cup S)$.

One may assume that the coloring c of $Q_1(p)$ is defined as follows: $c(s_i) = i$ for all $1 \leq i \leq p - 1$. For all $Q \in \mathcal{P}(p)$, put $c(v_Q) = p$ and $c(v_\emptyset) = p + 1$.

Let x_Q and $x_{Q'}$ be two vertices of K with $|Q| > |Q'|$. We have $|c(N[x_Q]) \cap S| > |c(N[x_{Q'}]) \cap S|$. The vertex x_\emptyset has not neighbors in S , then it is distinguished from any vertex x_Q with $Q \in \mathcal{P}(p)$ and $Q \neq \emptyset$. Moreover, the vertex x_\emptyset is used to distinguish s_j for $j = 1, 2$ and $x_Q \in K$ with $Q \in \mathcal{P}(p)$ and $Q \neq \emptyset$. \square

For the graph $Q_2(p)$ we have:

Proposition 18. *For $p \geq 2$, we have $\chi_{rlid}(Q_2(p)) = \omega(Q_2(p)) + 1$.*

Proof. Let c be a $rlid$ -coloring of $Q_2(p)$. For any vertex $s_i \in S$ with $i = 1, \dots, p-1$, we have $c(s_i) \notin c(V \setminus \{s_1, \dots, s_{p-1}\})$. Otherwise, we get $c(N[v_i]) = c(N[v_p])$.

Moreover, for $1 \leq i, j \leq p-1$ with $i \neq j$, we have $c(s_i) \neq c(s_j)$. Otherwise, we obtain $c(N[v_i]) = c(N[v_j])$.

One may assume that the coloring c of $Q_2(p)$ is defined by: for all $s_i \in S$, put $c(s_i) = i$ with $i = 1, \dots, p-1$. For any vertex $v_i \in K$, $c(v_i) \notin \{1, \dots, p-1\}$ for all $i = 1, \dots, p-1$.

If $|c(K)| = 1$ then $c(N[s_i]) = c(N[v_i])$ for any $i = 1, \dots, p-1$. Thus $|c(K)| \geq 2$, which give $\chi_{rlid}(Q_2(p)) \geq \omega(Q_2(p)) + 1$.

To conclude, observe that the coloring c defined by $c(s_i) = i$ for all $i = 1, \dots, p-1$, $c(v_i) = p$ for all $i < p$ and $c(v_p) = p+1$. Then c is $(p+1)$ - $rlid$ -coloring of $Q_1(p)$. \square

6 Discussion and open problems

We have introduced the notion of *relaxed locally identifying coloring* of graphs and we were interested in giving the minimum of colors used in a relaxed locally identifying coloring χ_{rlid} .

In Section 1, we have considered the graph containing twins and given a lower and an upper bound for χ_{rlid} for this graphs depending on χ_{rlid} of the maximal twin-free subgraph associated to this graph.

In Section 3, we have studied the problem of complexity of $rlid$ -coloring. We have shown that the problem of deciding that χ_{rlid} equals 3 is NP -complete for 2-degenerate planar graphs, and polynomial for bipartite graphs.

In Section 4, we have given a graph for which the lower bound of Theorem 1 is tight. We characterized graphs G satisfying $\chi_{rlid}(G) = n$. We have also given a lower bound of χ_{rlid} and we exhibited a family of graphs for which this bound is tight.

In Theorem 15, we proved that for a *Split* graph G , $\chi_{rlid}(G) \leq \omega(G) + 2$. Since we didn't find any split graph attaining this bound, we think that it will be improved. We propose the following conjecture:

Conjecture 19. *If G is a free-twin split graph, then $\chi_{rlid}(G) \leq \omega(G) + 1$.*

The notion of *relaxed locally identifying chromatic number* for finite graphs is new. It might be interesting to investigate other graphs as cographs, planar graphs, outerplanar graphs, line graphs, interval graphs, for graphs with a given maximum degree.

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